Global Optimization Methods based on Interval analysis

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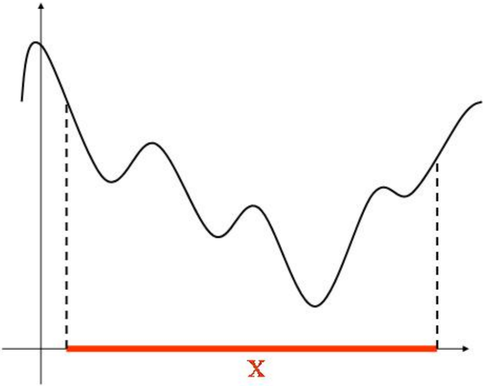
Introduction to Deterministic Global Optimization



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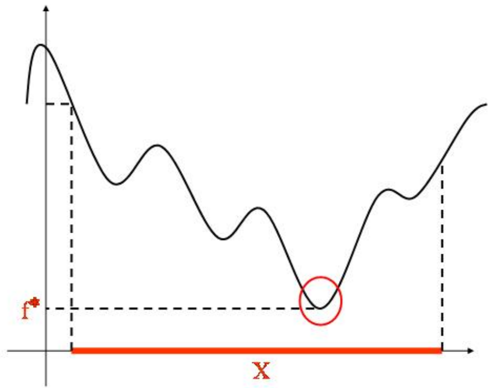

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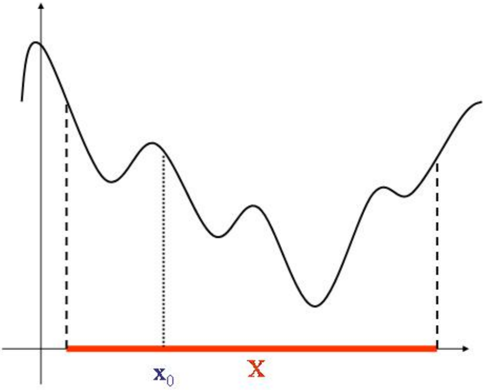
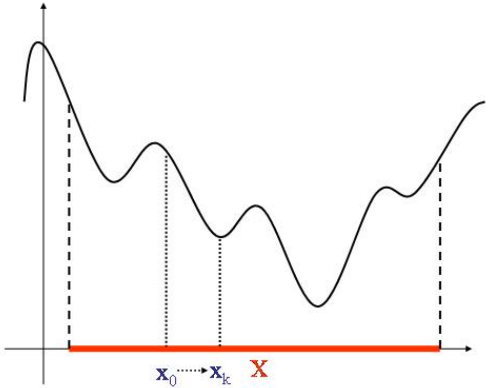
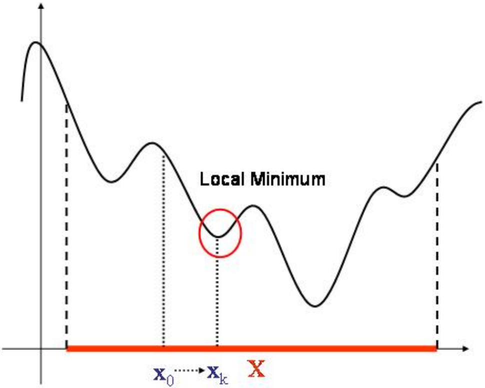



$x_0 \dashrightarrow x_k$
X


Local Minimum
x0 → xk
X

Introduction to Deterministic Global Optimization

- Multistart Method
- Metaheuristic Methods
  - Taboo Research, (Glover and Hansen),
  - VNS, (Mladenovitch and Hansen),
  - Kangourou Method...
- Stochastic Global Optimization Methods
  - Simulated Annealing,
  - Genetic Algorithms,
  - Evolutionary Algorithms...
- Deterministic Global Optimization Methods
  - Particular structure of problems:
    - Convex functions + Theory,
    - Linear programs: Simplex Algorithm (Danzig)
    - Quadratic programs: (Sherali, Audet, Hansen et al.)...,
  - More General Problems $\Longrightarrow$ Branch and Bound Techniques
    - Difference of convex or monotonic functions, (Horst and Tuy),
    - Interval analysis (Ratsheck, Rokne, E. Hansen)...

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Interval Analysis

Let $X = [x^L, x^U]$ and $Y = [y^L, y^U]$ 2 intervals.
Moore (1966) defines the interval arithmetic as follows:

$$\left\{\begin{array}{l} [x^L, x^U] + [y^L, y^U] = [x^L + y^L, x^U + y^U] \\ [x^L, x^U] - [y^L, y^U] = [x^L - y^U, x^U - y^L] \\ [x^L, x^U] \times [y^L, y^U] = \big[\min\{x^L y^L, x^L y^U, x^U y^L, x^U y^U\}, \\ \qquad\qquad\qquad\qquad \max\{x^L y^L, x^L y^U, x^U y^L, x^U y^U\}\big] \\ [x^L, x^U] \div [y^L, y^U] = [x^L, x^U] \times [\frac{1}{y^U}, \frac{1}{y^L}] \text{ if } 0 \notin [y^L, y^U]. \end{array}\right.$$

Remark

Subtraction and division are not the inverse operations of addition and respectively multiplication.

Difficulties:

$\div 0 \Longrightarrow$ extended interval arithmetic, (E. Hansen).

Numerical errors $\Longrightarrow$ rounded interval analysis, (Moore).

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Some Properties of Interval Analysis and Inclusion Function

Property

For all $x \in X$ and $y \in Y$, one has: $x \star y \in X \star Y$, where $\star$ is $+, -, \times, \div$.

Property

Let A, B, C 3 intervals, therefore $A \times (B + C) \subseteq A \times B + A \times C$.

Property

Let Y_1, Y_2, Z_1, Z_2 4 intervals, if $Y_1 \subseteq Z_1$ and if $Y_2 \subseteq Z_2$ then $Y_1 \star Y_2 \subseteq Z_1 \star Z_2$ where $\star$ is $+, -, \times, \div$.

Definition

An inclusion function $F(X)$ of f over a box X is such that

$$f(X) := [\min_{x \in X} f(x), \max_{x \in X} f(x)] \subseteq F(X) = [F^L(X), F^U(X)]$$

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Natural Extension: an Inclusion Function

Theorem

The natural extension into interval of an expression of f over a box X is an inclusion function.

Example

Let $f(x) = x^2 - x + 1$ and $x \in X = [0,1]$

Inclusion functions:

- $F_1(X) = X^2 - X + 1 = [0,1]^2 - [0,1] + [1,1] = [0,2]$,
- $F_2(X) = X(X-1) + 1 = [0,1]([0,1]-1) + [1,1] = [0,1] \times [-1,0] + [1,1] = [0,1]$,
- $F_3(X) = \left(X - \frac{1}{2}\right)^2 + \frac{3}{4} = \left[-\frac{1}{2}, \frac{1}{2}\right]^2 + \frac{3}{4} = \left[\frac{3}{4}, 1\right]$,

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Rounded Interval Analysis

Let $X = [a, b]$ and $Y = [c, d]$ 2 intervals.
Moore defines also the rounded interval arithmetic as follows:

$$\begin{cases} [a,b]+[c,d]=[\underline{a+c},\overline{b+d}] \\ [a,b]-[c,d]=[\underline{a-d},\overline{b-c}] \\ [a,b]\times[c,d]=[\min\{\underline{ac},\underline{ad},\underline{bc},\underline{bd}\}, \\ \qquad\qquad\qquad\quad \max\{\overline{ac},\overline{ad},\overline{bc},\overline{bd}\}] \\ [a,b]\div[c,d]=[a,b]\times[\frac{1}{d},\frac{1}{c}] \text{ if } 0\notin[c,d]. \end{cases}$$

Where $\underline{a}$, resp. $\overline{a}$, represents the nearest under, resp. over, floating point representation of the real x

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Extended Interval Analysis

Let $X = [a, b]$ and $Y = [c, d]$ 2 intervals.
E. Hansen defines the extended interval arithmetic for the division X/Y with $0 \in Y$, as follows:

$$X/Y = \begin{cases} [b/c, +\infty], \text{ if } b \leq 0 \text{ and } d = 0, \\ [-\infty, b/d] \cup [b/c, +\infty], \text{ if } b \leq 0 \text{ and } c < 0 < d, \\ [-\infty, b/d], \text{ if } b \leq 0 \text{ and } c = 0, \\ [-\infty, +\infty], \text{ if } a < 0 < b, \\ [-\infty, a/c], \text{ if } a \geq 0 \text{ and } d = 0, \\ [-\infty, a/c] \cup [a/d, +\infty], \text{ if } a \geq 0 \text{ and } c < 0 < d, \\ [a, b] \pm [-\infty, +\infty] = [-\infty, +\infty] \\ [a/d, +\infty], \text{ if } a \geq 0 \text{ and } c = 0, \end{cases}$$

For the addition and the substraction:

$$\begin{cases} [a, b] + [-\infty, d] = [-\infty, b + d] \\ [a, b] + [c, +\infty] = [a + c, +\infty] \\ [a, b] \pm [-\infty, +\infty] = [-\infty, +\infty] \\ [a, b] - [-\infty, d] = [a - d, +\infty] \\ [a, b] - [c, +\infty] = [-\infty, b - c] \end{cases}$$

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Outline

Inclusion Functions based on Taylor's Expansions: Univariate Case

Let f be a univariate differentiable function, and x, y and ξ, 3 variables of X an interval of $\mathbb{R}$.

$$f(x) = f(y)+(x-y)f'(y)+\frac{(x-y)^2}{2}f''(y)+\ldots+\frac{(x-y)^n}{n!}f^{(n)}(\xi)$$

Let denote $F^{(n)}(X)$ an enclosure of $f^{(n)}(\xi)$ over X (computed with an interval automatic differentiation tool).

Hence,

$$f(x) \in f(y)+(x-y)f'(y)+\frac{(x-y)^2}{2}f''(y)+\ldots+\frac{(x-y)^n}{n!}F^{(n)}(X)$$

2 inclusion functions:

- $T_1(y,X) = f(y) + (X-y)F'(X)$
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Baumann Centered Forms: Univariate Case

Optimal Baumann center $\underline{c}_B$ for the best lower bound for T_1:

$$\underline{z}_B := T_1^L(\underline{c}_B, X) = \max_{y \in X} T_1^L(y, X) = \big(f(y) + (X - y)F'(X)\big)^L$$

Optimal Baumann center $\overline{c}_B$ for the best upper bound for T_1:

$$\overline{z}_B := T_1^U(\overline{c}_B, X) = \min_{y \in X} T_1^U(y, X) = \big(f(y) + (X - y)F'(X)\big)^U$$

Baumann in 1988 gives analytical solution for $\underline{c}_B$ (and $\underline{c}_B$).

$$\underline{c}_B := \frac{x^L (F')^U(X) - x^U (F')^L(X)}{(F')^U(X) - (F')^L(X)}$$

if $0 \notin F'(X)$, else monotony case.
Easy to generalize to multivariate differentiable functions.

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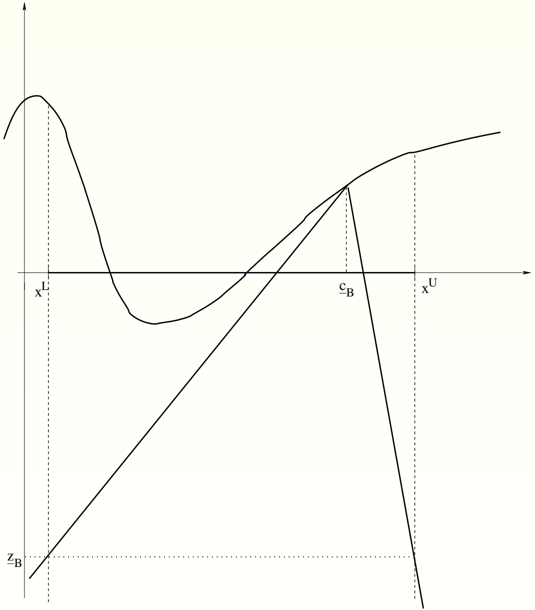
Baumann in 1988 gives analytical solution for $\underline{c}_B$ (and $\underline{c}_B$).

$$\underline{c}_B := \frac{x^L (F')^U(X) - x^U (F')^L(X)}{(F')^U(X) - (F')^L(X)}$$

if $0 \notin F'(X)$, else monotony case.
Easy to generalize to multivariate differentiable functions.

Example of Baumann Lower Bounds: Univariate Case



Linear Boundary Value Form of Neumaier and Hansen et al. 90-92: Univariate Case

Start with $f(x) \in f(y) + (x-y)F'(X), \forall (x,y) \in X^2$

Case when $0 \in F'(X)$ (else it is obvious):

2 affine underestimations:

- $f(x) \geq f(x^L) + (x - x^L)(F')^L(X), \forall x \in X,$
- $f(x) \geq f(x^U) + (x - x^U)(F')^U(X), \forall x \in X,$

Therefore, the intersection is a minorant of f over X:

$$z_{lbvf} = \frac{(F')^U(X)f(x^L) - (F')^L(X)f(x^U)}{(F')^U(X) - (F')^L(X)} + \frac{(x^U - x^L)(F')^L(X)(F')^U(X)}{(F')^U(X) - (F')^L(X)}$$

Same think for constructing a majorant.

Problem for a generalization to the multivariate differentiable case.

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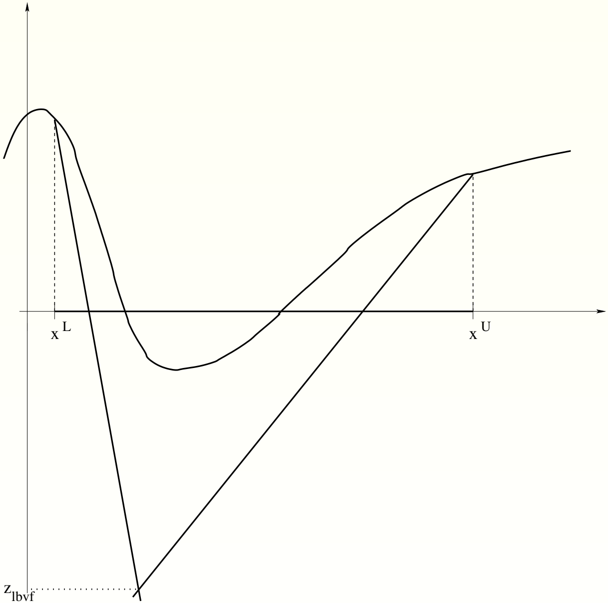
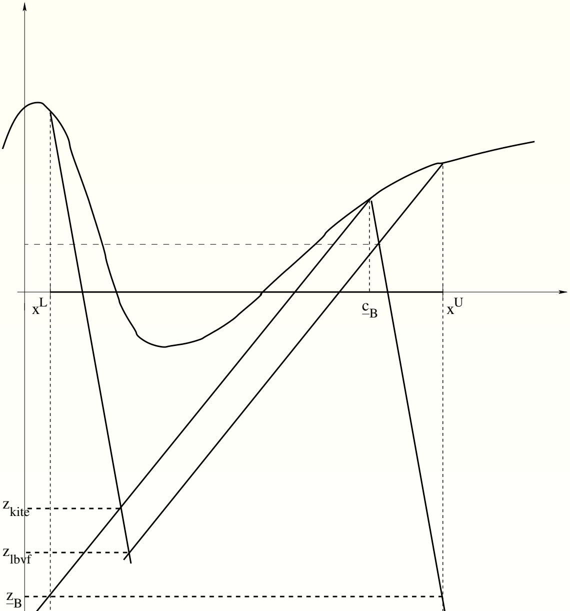


Example of Kite Algorithm of Lagouanelle, Csendès et al.



Comparison between Baumann and Linear Boundary Value Forms: Univariate Case

Notations: $X = [a, b]$ and $F'(X) = [L, U]$, with $L < 0 < U$.

Baumann Form:

$$\underline{c}_B = \frac{aU - bL}{U - L} \text{ and } \underline{z}_B = f(\underline{c}_B) + \frac{(b-a)LU}{U-L}$$

Linear Boundary Value Form:

$$z_{lbvf} = \frac{U}{U-L} f(a) + \frac{-L}{U-L} f(b) + \frac{(b-a)LU}{U-L}$$

Comparison between:

$$f(\underline{c}_B) <>? \frac{U}{U-L} f(a) + \frac{-L}{U-L} f(b)$$

- $>$ BF produces the best lower bound,
- $<$ LBVF produces the best lower bound,
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Example of the use of T_1, T_B and $LBVF$ Methods: Univariate Case

$$f(x) = x^2 - x, x \in X = [0,2], \text{ and } \min_{x \in [0,2]} f(x) = -\frac{1}{4}$$

One has:

$$F(X) = [0,2]^2 - [0,2] = [-2,4]$$

and

$$G(X) = 2X - 1 = [-1,3]$$

$$T_1(X) = f(1) + ([0,2] - 1) \times [-1,3] = [-3,3]$$

$$\underline{c_B} = \frac{1}{2}, T_B^L(\frac{1}{2}, X) = -\frac{7}{4}$$

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$$f(\underline{c_B}) = f\left(\frac{1}{2}\right) = -\frac{1}{4} < \frac{2 \times f(0) - 0 \times f(2)}{2 - 0} = 0$$

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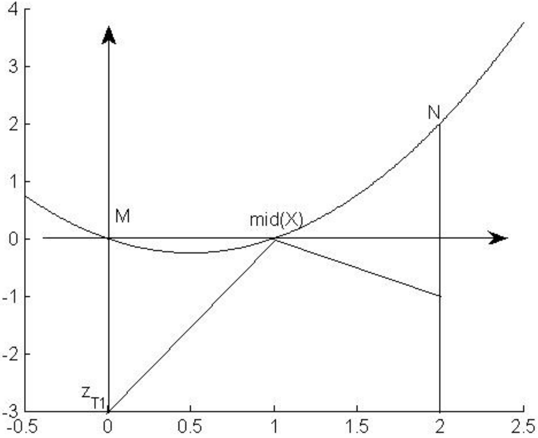
Therefore *LBVF* give the best lower bound:

$$z_{lbvf} = -1$$

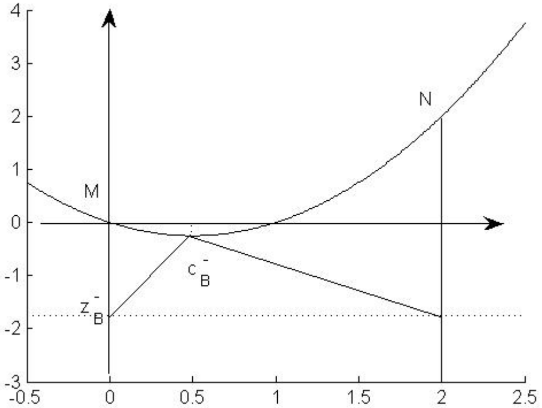
Example of the use of T_1 at $mid(X)$: Univariate Case

$$f(x) = x^2 - x, x \in X = [0, 2], T_1^L(X) = -3$$



Example of the use of T_1 at the Baumann center c_B^-: Univariate Case

$$f(x) = x^2 - x, x \in X = [0, 2], T_B(X) = -1.75$$



Example of the use of *LBVF*: Univariate Case

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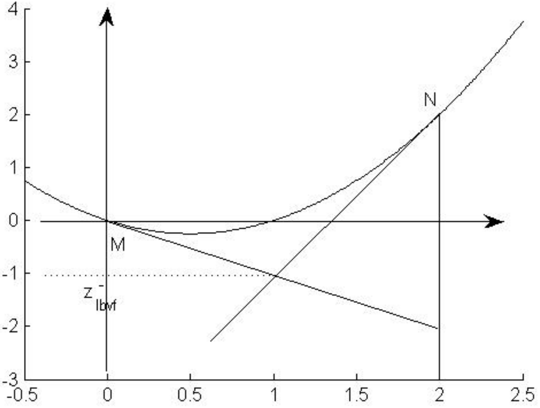
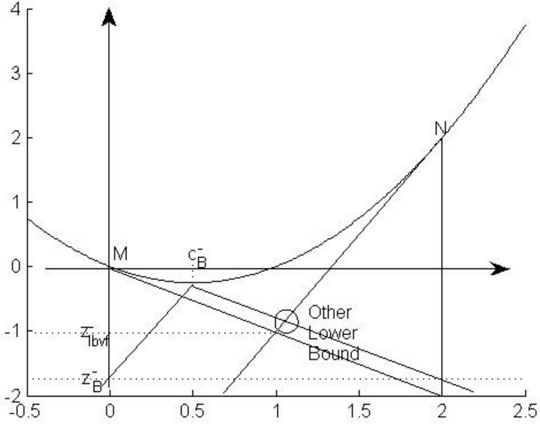


Example of the use of the Kite Algorithm

$$f(x) = x^2 - x, x \in X = [0, 2], KITE(X) > -1$$



Translation based Method for polynomial functions: Univariate Case

Idea: Compute bounds of f over X by translating the box X.

$$X \longrightarrow X + \mu \text{ implies modifications of } f$$

$$\begin{aligned} p(x) &= \sum_{k=0}^{n} a_k x^k = \sum_{j=0}^{n} a_j((x+\mu)-\mu)^j \\ &= \sum_{j=0}^{n} (x+\mu)^j \sum_{k=0}^{n-j} a_{k+j} \binom{k+j}{j} (-\mu)^k \\ &= \sum_{j=0}^{n} f_j(\mu)(x+\mu)^j, \end{aligned}$$

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Example for Translation based Method: Univariate Case

Example

Let $p(x) = 7x^4 - 5x^3 + 4x^2 + 3x + 2$ and $x \in X = [0, 10]$

Lower bounds: (global optimum about 3).

- $NE^L(X) = -4.9\ 10^3$,
- $H^L(X) = -4.57\ 10^3$,
- $T_1^L(X) = -1.37\ 10^5$,
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Inclusion Functions based on Taylor's Expansions: Multivariate Case

Let f be a multivariate differentiable function, and x and y, 2 variables of X an interval of $\mathbb{R}^n$.

2 inclusion functions:

- $T_1(y,X) = f(y) + (X-y).G(X)$
  $\qquad\qquad = f(y) + \sum_{i=1}^{n}(X_i - y_i).G_i(X)$
- $T_2(y,X) = f(y) + (X-y)f'(y) + \dfrac{1}{2}(X-y)^T.H(X).(X-y)$

$G(X)$ represents the enclosure of the gradient and $H(X)$ the enclosure of the Hessian matrix over X.

Notations and Assumptions

Notations:

- $X = (X_1, \ldots, X_n)$, where $X_i \subseteq \mathbb{R}$,
- $X_i = [a_i, b_i]$,
- $\dfrac{\partial f}{\partial x_i}(x) \in [L_i, U_i]$,
- f is define from X to $\mathbb{R}$.

Assumptions:

- $L_i < 0 < U_i$ (else monotonicity case),
- f is one time differentiable over X.

Baumann Centered Form in the Multivariate Case

Easy to generalize to all the variables are separated:
Optimal Baumann center $\underline{c_B}$ for the best lower bound for T_1:

$$\underline{z_B} := T_1^L(\underline{c_B}, X) = \max_{y \in X} T_1^L(y, X) = (f(y) + (X - y).G(X))^L$$

Baumann (1988) gives analytical solution for $\underline{c_B}$ (and $\overline{c_B}$).

$$\underline{(c_B)_i} := \frac{a_i U_i - b_i L_i}{U_i - L_i}, \forall i \in \{1, \dots, n\}.$$

Hence,

$$\underline{z_B} = f(\underline{c_B}) + \sum_{i=1}^{n} \frac{(b_i - a_i) L_i U_i}{U_i - L_i}$$

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Extension of *LBVF* to the Multivariate Case: Admissible Simplex Method

n variables $\Longrightarrow 2^n$ affine underestimations.

Choice of $n+1$ of them (among 2^n).

Idea: Construction of an Admissible Path from S to $\overline{S}$ (opposite vertex of S).

- Admissible Simplex:

Figure: Example of an admissible simplex with 3 variables

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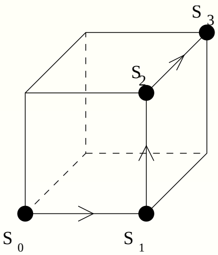

Figure: Example of an admissible simplex with 3 variables

Admissible Simplex Form

S_k denotes a vertex of the hypercube X.

$$f(x) \geq f(S_k) + \sum_{i \in I_k} (x_i - a_i) L_i + \sum_{j \in J_k} (x_j - b_j) U_j, for all x \in X,$$

where $I_k \subset N = \{1, 2, \ldots, n\}$, $J_k = N - I_k$ and $j \in I_k$ iff $(S_k)_j = a_j$ (else $j \in J_k$).

Construction of an Admissible Simplex:
Find an admissible set: of vertices $S_0, S_1, \ldots, S_n$, means that the intersection of their corresponding hyperplane Π_k => lower bound of f over X.

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where $I_k \subset N = \{1, 2, \ldots, n\}$, $J_k = N - I_k$ and $j \in I_k$ iff $(S_k)_j = a_j$ (else $j \in J_k$).

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Find an admissible set: of vertices $S_0, S_1, \ldots, S_n$, means that the intersection of their corresponding hyperplane $\Pi_k =>$ lower bound of f over X.

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This set of vertices is admissible iff

$$\alpha_0 = H_{l_1} \leq 1, \alpha_1 = H_{l_1} - H_{l_2} \geq 0, \ldots, \alpha_{l_{n-1}} = H_{l_{n-1}} - H_{l_n} \geq 0, \alpha_n =$$

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Baumann Form versus Admissible Simplex Method: Multivariate Case

Proposition

The Baumann center is inside each admissible simplex.

Proposition

Comparison:

$$f(\underline{c}_B) <>? \sum_{i=0}^{n} \alpha_i f(S_i)$$

- $>$ *BF produces the best lower bound,*
- $<$ *ASF produces the best lower bound,*
- $=$ *equality occurs.*

Property

For computing lower bounds:

- *f convex $\Longrightarrow$ ASF.*
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Affine and Quadratic Forms

Difficulty of interval analysis: occurrences of the same variables.

For example: $x - x$, with $x \in [0, 1]$ yields to $[0, 1] - [0, 1] = [-1, 1]$.

Idea: Keep affine information during the computations.

$$GQF(X) \subseteq QF(X) \subseteq AF_2(X) \subseteq AF_1(X) = AF(X)$$

Example

Let $f(x, y) = x^2y - xy^2$ and $(x, y) \in X^2 = [-1, 3]^2$

Inclusion functions:

- $AF(X) = AF_1(X) = [-44, 44]$.
- $AF_2(X) = QF(X) = [-40, 40]$.
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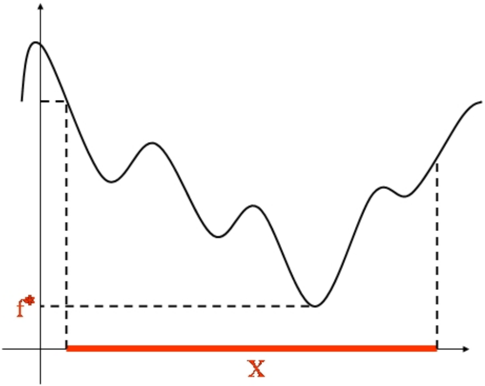
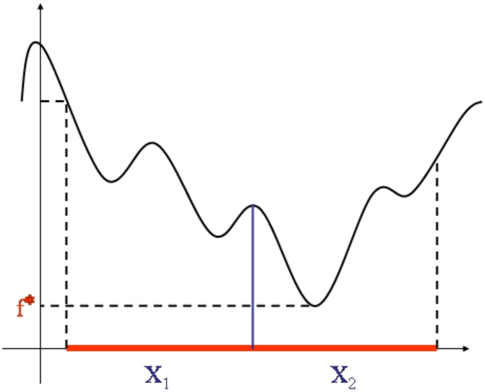
Outline

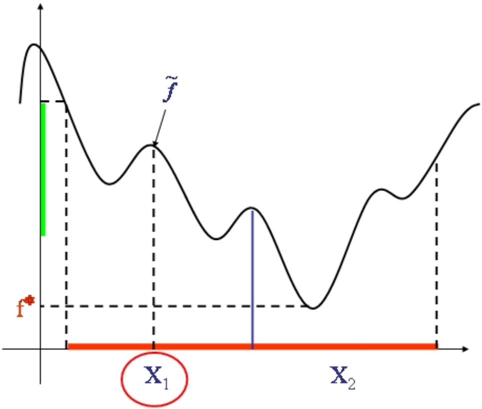
Branch and Bound Algorithm for Continuous Optimization Problems: Unconstrained Case



Branch and Bound Algorithm for Continuous Optimization Problems: Unconstrained Case



Interval Branch and Bound Algorithm for Continuous Optimization Problems: Unconstrained Case



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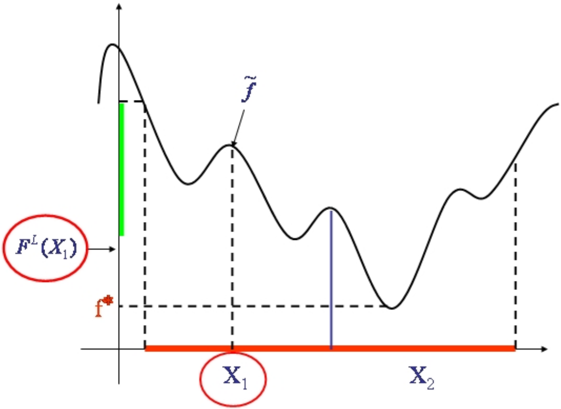

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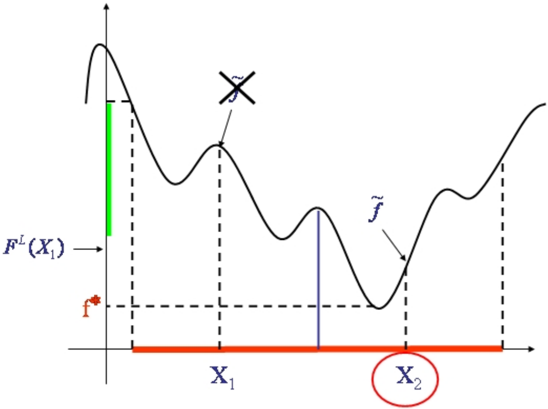

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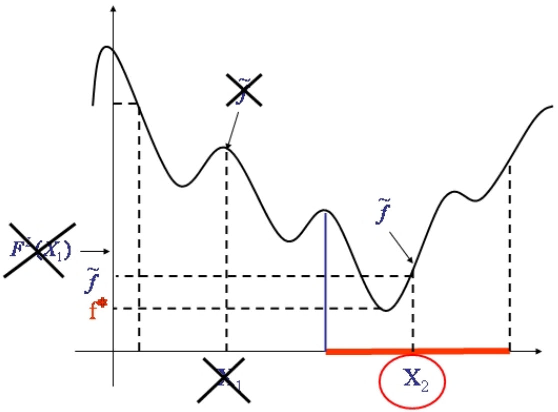

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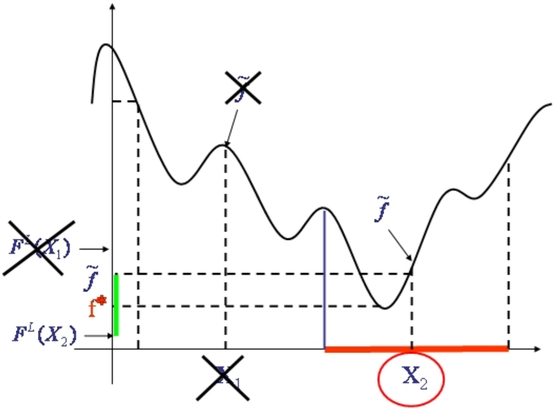

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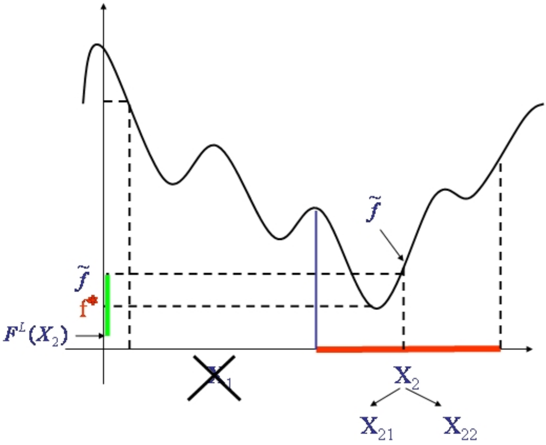

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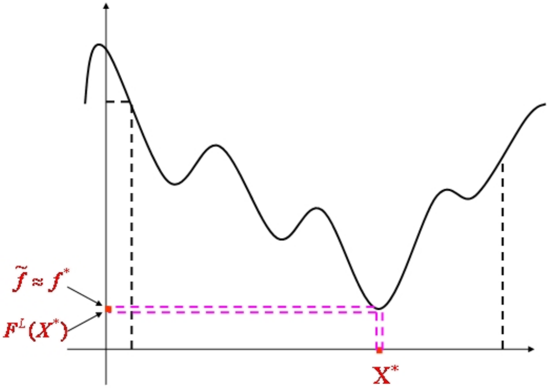

Principle of an Interval Branch and Bound Algorithm due to Ichida-Fujii: Unconstrained Case

- Choice and Subdivision of the box X, (in 2 parts for each iteration) $\Longrightarrow$ list of possible solutions,
- (Reduction of the sub-boxes, by using a monotonicity tests ...),
- Computation of bounds of a differentiable function f over a sub-boxe, (inclusion functions)
- Elimination of the sub-boxes which cannot contain the global optimum: $F^L(X) > \tilde{f}$; where $\tilde{f}$ denotes the current solution,
- STOP when accurate enclosures of the optimum are obtained.

Interval Branch and Bound Algorithm: Choice, Subdivision and Stopping Criteria

- Choice a box in a list:
  - Largest box,
  - Oldest box,
  - The box which has the lowest lower bound.
  - Other Heuristics, see Csendes et al.
- Bisection of a box:
  - in two equal parts following the largest edge,
  - in two parts following the Baumann center,
  - in two equal parts following the largest $D_i = w(G_i(X))w(X_i)$
  - Multisection methods, see Csendes et al., and Lagouanelle-Soubry (JOGO 2004) for theoritical results on optimal multisection techniques.
- Stopping criteria:
  - the smallest lower bound of all the boxes in the list is sufficiently closed to $\tilde{f}$,
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Numerical Examples: Baumann Form vs Admissible Simplex Form

Standard-PC computer with an 1.8GHz AMD Athlon Processor and 256Mb RAM using a Fortran-90 compiler.

All the computations, even the floating-point operations, are performed using rounded interval analysis.

14 differentiable functions from the literature, as for examples:

- $f_1(x) = 1 + (x_1^2 + 2)x_2 + x_1 x_2^2, x_1 \in [1, 2], x_2 \in [-10, 10]$ $\epsilon = 10^{-8}$, $f^* = -3.5, x^* = (2, -1.5)$.
- $f_2(x) = 2x_1^2 - 1.05x_1^4 + x_2^2 - x_1 x_2 + \frac{1}{6}x_2^6, \forall x_i \in [-2, 4]$ $\epsilon = 10^{-8}$, $f^* = -239.696, x^* = (4, 1.115)$.
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- $f_3(x) = (x_1 - 2x_2 - 7)^2 + (2x_1 + x_2 - 5)^2, x_1 \in [-2.5, 3.5], x_2 \in [-1.5, 4.5]$ $\epsilon = 10^{-8}$, $f^* = 0.45, x^* = (3.4, -1.5)$.

Where, f^* represents the optimal value and x^* a corresponding optimizer.

Numerical Results (1)

Pbs	*T1*		*TB*		*AS*		*MIXED*	
	Its	time(s)	Its	time(s)	Its	time(s)	Its	time(s)
f_1	167	0.16	133	0.11	131	0.11	132	0.11
f_2	87	0.11	87	0.11	84	0.16	86	0.11
$-f_2$	116	0.17	109	0.11	84	0.16	110	0.11
f_3	124	0.11	107	0.11	101	0.11	101	0.11
f_4	5503	2.64	3847	1.92	3734	1.92	3731	2.15
f_5	735	0.39	364	0.22	364	0.33	364	0.33
$-f_5$	148	0.21	127	0.22	128	0.22	125	0.17
f_6	2464	1.27	1585	0.77	1136	0.60	1183	0.66
f_7	13856	4.50	9908	3.46	9778	3.63	9587	3.84
f_8	266973	2649.10	112841	496.2	57023	116.33	57023	116.87
f_9	1222	0.49	753	0.27	691	0.27	687	0.28
f_{10}	1270	0.39	813	0.28	693	0.27	689	0.28
f_{11}	38048	68.22	26894	42.89	21583	27.91	21583	28.17
$-f_{11}$	265	0.17	183	0.11	180	0.17	178	0.22

Table: Numerical Results with $f(c)$

where $c = mid(X)$.

Numerical Results (2)

Pbs	$T1 + f(c')$		$TB + f(c_B^-)$		$AS + f(c')$		$MIXED + f(c')$ or $f(c_B^-)$	
	Its	time(s)	Its	time(s)	Its	time(s)	Its	time(s)
f_1	95	0.05	72	0.05	73	0.05	71	0.06
f_2	41	0.05	39	0.05	37	0.06	37	0.05
$-f_2$	64	0.05	56	0.06	84	0.16	53	0.05
f_3	71	0.05	57	0.06	54	0.06	54	0.06
f_4	5503	2.47	3847	1.81	3734	1.92	3731	1.98
f_5	453	0.16	221	0.05	213	0.05	213	0.05
$-f_5$	49	0.01	25	0.04	26	0.01	26	0.01
f_6	1330	0.55	840	0.33	597	0.33	596	0.33
f_7	13856	3.95	9908	2.69	9778	3.19	9587	3.13
f_8	266885	2744.02	112769	492.73	57023	113.92	56987	116.55
f_9	1222	0.22	753	0.16	691	0.22	687	0.21
f_{10}	1270	0.32	811	0.06	693	0.17	689	0.11
f_{11}	38048	67.44	26894	42.62	21583	27.63	21583	28.23
$-f_{11}$	217	0.05	135	0.01	132	0.05	130	0.01

Table: Numerical Results with $f(c')$ or $f(c_B^-)$

where $c' = mid(X)$ but $c'_i = a_i$ if $L_i \geq 0$ and else $c'_i = b_i$.

Numerical Results (3)

Pbs	$MIXED + f(c)$			$MIXED + f(c')$ or $f(c_B^-)$		
	Nb AS	Nb Baumann	Nb equality	Nb AS	Nb Baumann	Nb equality
f_1	193	27	44	116	11	15
f_2	85	10	77	52	4	18
$-f_2$	109	31	72	81	5	20
f_3	167	0	35	91	0	17
f_4	6696	763	3	6696	763	3
f_5	508	60	160	332	19	75
$-f_5$	26	15	209	26	15	11
f_6	2024	176	166	332	19	75
f_7	13624	2974	2576	13624	2974	2576
f_8	107818	93	6135	107748	93	6133
f_9	1198	58	118	1198	58	118
f_{10}	1088	108	182	1088	108	182
f_{11}	32250	57	10859	32250	57	10859
$-f_{11}$	155	79	122	155	79	26

Table: Number of computed lower bounds

Numerical Results: Affine and Quadratic Arithmetic

Functions	Initial Domain of Research
$f_1(x) = 1 + (x_1^2 + 2)x_2 + x_1 x_2^2$	$X = [1, 2] \times [-10, 10]$ $f_1^* = -3.5$
$f_2(x) = x_1^3 x_2 + x_2^2 x_3 x_4^2 - 2x_5^2 x_1 + 3x_2 x_4^2 x_5$	$X = [-10, 10]^5$ $f_2^* = -1042000$
$f_3(x) = (x_1 - 2x_2 - 7)^2 + (2x_1 + x_2 - 5)^2$	$X = [-2.5, 3.5] \times [-1.5, 4.5]$ $f_3^* = 0.45$
$f_4(x) = [1 + (x_1 + x_2 + 1)^2$ $(19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2^2)]$ $\times[30 + (2x_1 - 3x_2)^2$ $(18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2)]$	$X = [-2, 2]^2$ **Golstein Price function** $f_4^* = 3$
$f_5(x) = (x_1 - 1)(x_1 + 2)(x_2 + 1)(x_2 - 2)x_3^2$	$X = [-2, 2]^3$ $f_5^* = -36$
$f_6(x) = 4x_1^2 - 2x_1x_2 + 4x_2^2 - 2x_2x_3 + 4x_3^2 - 2x_3x_4$ $+4x_4^2 + 2x_1 - x_2 + 3x_3 + 5x_4$	$X = [-1, 3] \times [-10, 10] \times [1, 4] \times [-1, 5]$ $f_6^* = 5.77$
$f_7(x) = x_1^3 x_2 + x_2^2 x_3 x_4^2 - 2x_5^2 x_1 + 3x_2 x_4^2 x_5$ $-\frac{1}{6} x_5^5 x_4^3 x_3^2$	$X = [-10, 10]^5$ $f_7^* = -1667708716.3372$
$f_8(x) = 4x_1^2 - 2.1x_1^4 + \frac{1}{3}x_1^6 + x_1x_2 - 4x_2^2 + 4x_2^4$	$X = [-1000, 1000]^2$ **Ratschek function** $f_8^* = -1.03162845348366$
$f_9(x) = 4x_1^2 + 2x_2^2 - 5x_1^2 x_3 + 6x_3 x_4^2$ $-x_4^3 + 3x_4x_2 - x_3x_4 + 2x_1x_5 + 5x_5^2 x_2$	$X = [-10, 10]^5$ $f_9^* = -12001.8518518519$
$f_{10}(x) = 6.94x_1^4 + 0.96x_1^3 + 9.68x_1^2 + 4.16x_1$ $+7.53x_2^4 - 7.68x_2^3 + 8.21x_2^2 - 1.75x_2$ $-7.45x_1x_2 + 9.15x_1x_2^2 + 3.70x_1x_2^3 - 4.81x_1^2x_2$ $-3.06x_1^2x_2^2 - 0.79x_1^3x_2 - 0.18$	$X = [-50, 50]^2$ **A random polynomial function** $f_{10}^* = -0.61585524178857$

Table: Test Functions

Numerical Results: Affine and Quadratic Arithmetic (1)

Pbs	NE		T_1		AAR		AQR	
	CPU	ϵ_p	CPU	ϵ_p	CPU	ϵ_p	CPU	ϵ_p
f_1	49.9	10^{-5}	0.3	10^{-13}	0.4	10^{-13}	0.4	10^{-13}
f_2	0.6	10^{-8}	146.1	10^{-8}	1.0	10^{-8}	2.0	10^{-8}
f_3	13.9	10^{-7}	0.4	10^{-14}	0.5	10^{-14}	0.6	10^{-14}
f_4	612.0	1	9.9	10^{-12}	6.1	10^{-12}	3.2	10^{-12}
f_5	130.1	10^{-4}	1.0	10^{-12}	0.9	10^{-12}	1.2	10^{-12}
f_6	307.6	10^{-1}	135.8	10^{-12}	13.4	10^{-12}	15.8	10^{-12}
f_7	0.6	10^{-5}	—	—	17.1	10^{-4}	7.6	10^{-5}
f_8	3.9	10^{-2}	2.6	10^{-14}	8.5	10^{-14}	4.2	10^{-14}
f_9	3.7	10^{-1}	8.6	10^{-10}	3.3	10^{-10}	7.4	10^{-10}
f_{10}	64.8	10^{-3}	2.5	10^{-14}	8.1	10^{-14}	3.3	10^{-14}
avg	$118.7s$		$34.1s$ (without f_7)		$5.9s$		$4.5s$	

Table: Comparative Tests between Different Inclusion Functions

Numerical Results: Affine and Quadratic Arithmetic (1)

Pbs	*NE*		T_1		*AAR*		*AQR*	
	# its	# clst	# its	# clst	# its	# clst	# its	# clst
f_1	31501	16193	287	35	227	34	168	28
f_2	248	21	9148	21	314	25	269	36
f_3	26453	10597	335	48	280	42	206	38
f_4	117552	14231	6536	114	1849	39	1071	22
f_5	49432	20783	1832	130	919	81	698	64
f_6	47738	40580	10980	3096	5495	1057	4121	1309
f_7	257	37	—	—	3278	827	626	166
f_8	13125	7089	4431	91	3009	93	1553	31
f_9	10190	4770	2214	98	979	76	797	58
f_{10}	26419	19732	1724	31	1180	35	632	12
avg	32292	50400	4165	407	1753	231	1004	176
			(without f_7)					

Table: Comparative Tests between Different Inclusion Functions

Interval Branch and Bound Algorithm: Accelerating Subroutines

Notations: $X \subseteq \mathbb{R}^n$, we consider f over X, $G(X)$ is an enclosure of the gradient of f over X and $H(X)$ is an enclosure of the Hessian matrix of f over X.

- Monotonicity Test
  if $G_i(X)^L > 0$ then
  $X_i := [x^L, x^L]$
  if $G_i(X)^U < 0$ then
  $X_i := [x^U, x^U]$
  Reduction of the research on a face of X for all i.
- Convexity Test
  if $H_{ii}(X)^U < 0$ for a i then
  the hessian matrix cannot be semi-definite positive over X
  there is no stationary point in X
  X can be deleted.

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Interval Branch and Bound Algorithm: Interval Newton Step

Notations: $X \subseteq \mathbb{R}^n$, we consider f over X, $G(X)$ is an enclosure of the gradient of f over X and $H(X)$ is an enclosure of the Hessian matrix of f over X.

One Interval Newton Step:

1. Choose $x \in X$,
2. Solve $H(X)(x - Y) = \nabla f(x)$, denote Z the resulting enclosure of the solution Y
3. $X' := X \cap Z$.

Property

- *If ξ is a zero of ∇f then $\xi \in X'$.*
- *If $X' = \emptyset$ then ∇f does not have a zero in X.*
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Pruning Techniques based on T_1, Baumann and LBVF: Univariate Case

Idea: What can we do with a solution $\tilde{f}$?

The solution in X is over the line $y = \tilde{f}$.

$\Longrightarrow$ Discard some parts of the box.

Example:

Consider $f(x) = x^2 - x, x \in X = [0, 2]$ and $\tilde{f} = -\frac{1}{4}$ (the global minimum value)

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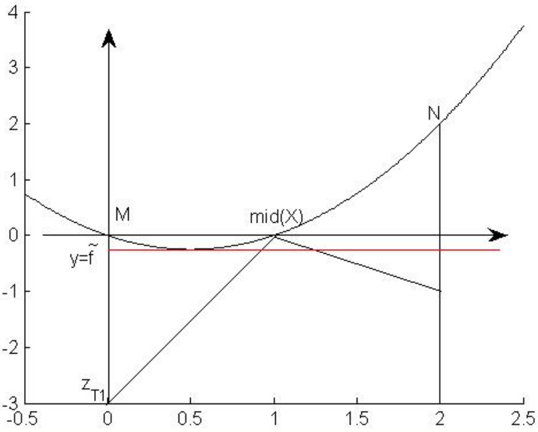
Consider $f(x) = x^2 - x, x \in X = [0, 2]$ and $\tilde{f} = -\frac{1}{4}$ (the global minimum value)

Example of Pruning Techniques based on T_1

$$f(x) = x^2 - x, x \in X = [0, 2], T_1^L(X) = -3$$



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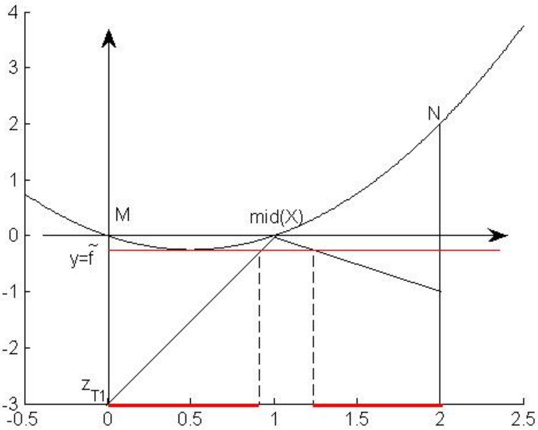
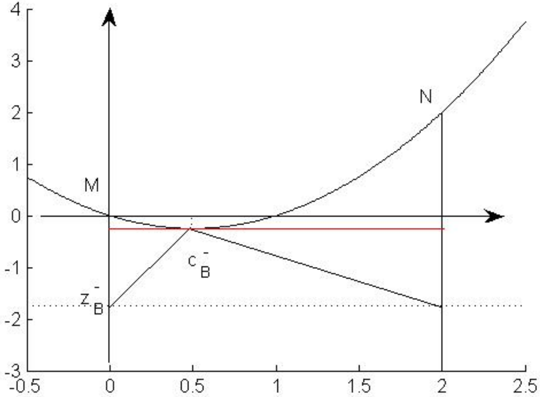


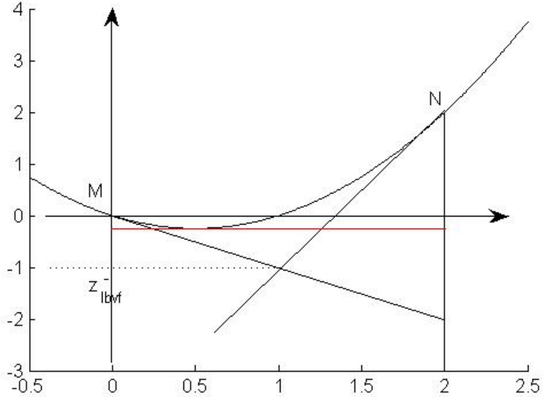
Example of Pruning Techniques based on T_B

$$f(x) = x^2 - x, x \in X = [0,2], T_B(X) = -1.75$$



Example of Pruning Techniques based on *LBVF*

$$f(x) = x^2 - x, x \in X = [0, 2], LBVF(X) = -1$$



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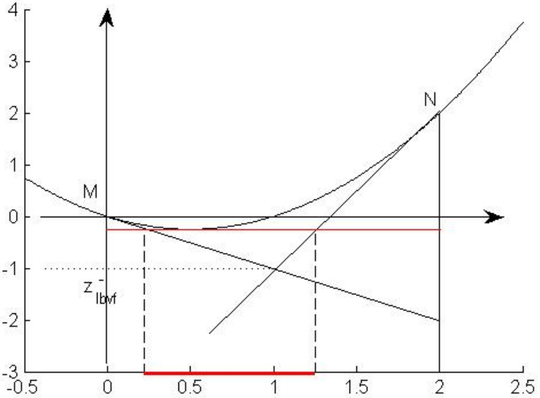

Outline

Principle of a Branch and Bound Algorithm for a problem with constraints

Notation:

$$\left\{\begin{array}{l}\min\limits_{x\in\mathbb{R}^n} f(x)\\ g_i(x)\leq 0\ \forall i\in\{1,\ldots,n_g\}\\ h_j(x)=0\ \forall j\in\{1,\ldots,n_h\}\end{array}\right.$$

- Choice and Subdivision of the box X, (in 2 parts by step): list of possible solutions,
- Reduction of the sub-boxes, by using a constraint propagation technique,
- Computation of bounds of the functions F, G_j, H_j on the sub-boxes, - inclusion functions -
- Elimination of the sub-boxes which cannot contain the global optimum: $F^L(X) > \tilde{f}$ or $G_i^L(X) > 0$ or $0 \notin H(X)$, where $\tilde{f}$ denotes the current solution,
- STOP when accurate enclosures of the optimum are obtained.

Propagation Techniques

$c(x) \in [a, b]$ is a contraint $\Longrightarrow$ implicit (or explicit) relations between the variables of the problem.

Idea: use some deduction steps for reducing the box X.

Linear case: if $c(x) = \sum_{i=1}^{n} a_i x_i$ then:

$$X_k := \left(\frac{[a, b] - \sum_{i=1, i \neq k}^{n} a_i X_i}{a_k} \right) \cap X_k, \text{ si } a_k \neq 0. \quad (1)$$

where k is in $\{1, \cdots, n\}$ and X_i is the i^{th} component of X.

Non-linear case: Idea (E. Hansen): one linearizes using T_1 (or T_2). Then one solve a linear system with interval coefficients.

Other Idea: construction of the calculus tree and propagation.

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$$X_k := \left(\frac{[a, b] - \sum_{i=1, i \neq k}^{n} a_i X_i}{a_k} \right) \cap X_k, \text{ si } a_k \neq 0. \quad (1)$$

where k is in $\{1, \cdots, n\}$ and X_i is the i^{th} component of X.

Non-linear case: Idea (E. Hansen): one linearizes using T_1 (or T_2). Then one solve a linear system with interval coefficients.
Other Idea: construction of the calculus tree and propagation.

Example of Propagation Technique based on the Calculus Tree

Let $c(x) = 2x_3x_2 + x_1$ and

$$c(x) = 3$$

where $x_i \in [1, 3]$ for all $i \in \{1, 2, 3\}$.

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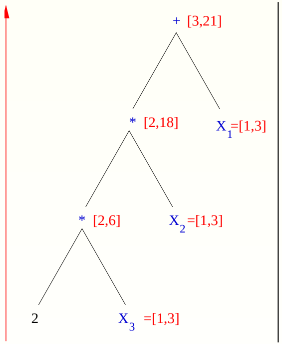

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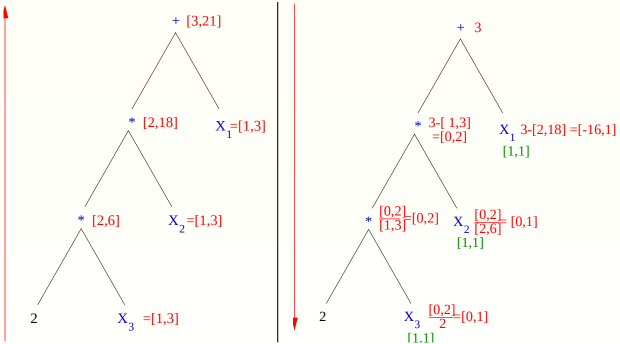

Algorithms for Mixed Problems

Continuous variables: real variables (dimensions of an electrical machines such as the diameter).

Discrete variables: integer (number of pair of poles of a machine), boolean (machine with or without slot), categorical variable (which kind of magnet is used).

For integer and boolean variables $\Longrightarrow$ relaxation for computing bounds + particular bisection technique and propagation.

For categorical variables $\Longrightarrow$ we introduce 4 particular algorithms with propagation and retro-propagation + properties about the bisection techniques.

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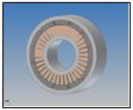
Outline

Direct and Inverse Problem of Design

1 – Direct Problem of Design



Direct and Inverse Problem of Design

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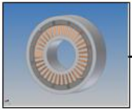

→ **Direct Solve of the Maxwell's Equations By Finite Element Methods**

Direct and Inverse Problem of Design

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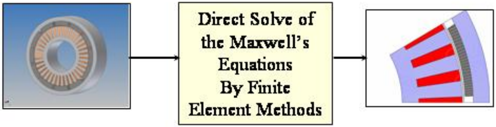
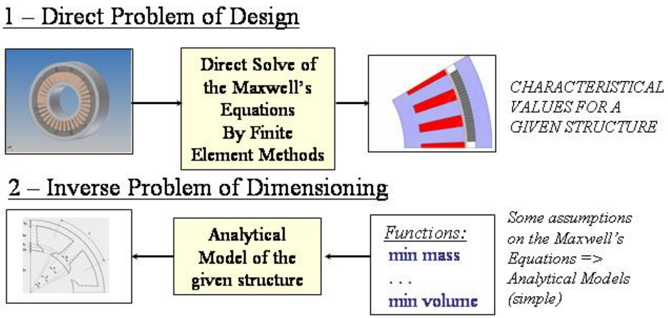


CHARACTERISTICAL VALUES FOR A GIVEN STRUCTURE

Direct and Inverse Problem of Design



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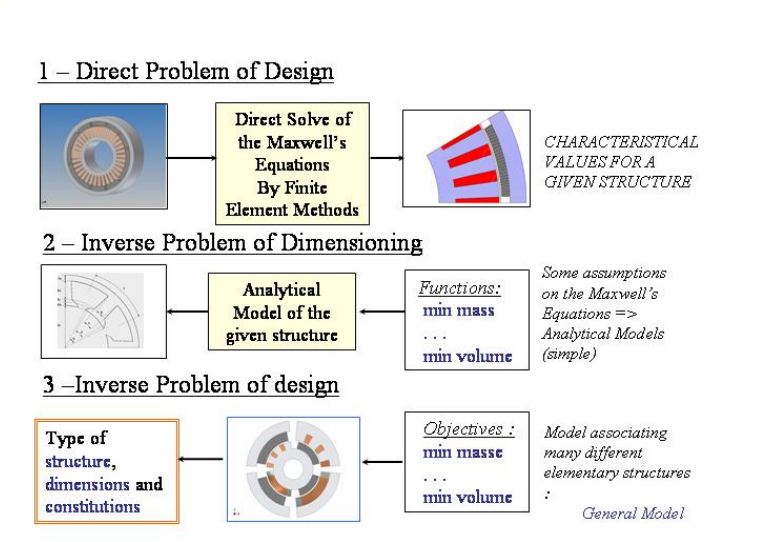

Mathematical Formulation

- Dimensioning Inverse Problem:

$$\begin{cases} \min\limits_{x \in \mathbb{R}^n} f(x) \\ g_i(x) \leq 0 \ \forall i \in \{1, \ldots, n_g\} \\ h_j(x) = 0 \ \forall j \in \{1, \ldots, n_h\} \end{cases}$$

- More General Inverse Problem of Design:

$$\begin{cases} \min\limits_{\substack{x \in \mathbb{R}^{n_r}, z \in \mathbb{N}^{n_e}, \\ \sigma \in \prod_{i=1}^{n_c} K_i, b \in B^{n_b}}} f(x, z, \sigma, b) \\ g_i(x, z, \sigma, b) \leq 0 \ \forall i \in \{1, \ldots, n_g\} \\ h_j(x, z, \sigma, b) = 0 \ \forall j \in \{1, \ldots, n_h\} \end{cases}$$

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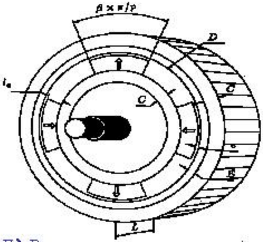
- **Criteria :**

$$
\begin{cases}
V_{ap} = \pi \dfrac{D}{\lambda}(D+E-e-l_a)(2C+E+e+l_a) \\
V_m = \pi \beta l_a \dfrac{D}{\lambda}(D-2e-l_a) \\
P_j = \pi \rho_{cu} \dfrac{D}{\lambda}(D+E)E_{ch}
\end{cases}
$$



Constraints :

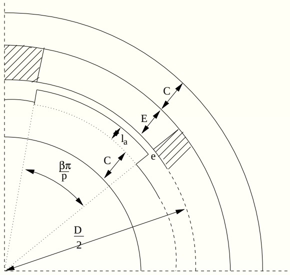
$$C_{em} = \frac{\pi}{2\lambda}(1-K_f)\sqrt{k_r \beta E_{ch} E}\, D^2 (D+E) B_e$$

$$E_{ch} = AJ_{cu} = k_r E J_{cu}^2, K_f \approx 1.5\, p \beta \frac{e+E}{D}, B_e = \frac{2 l_a P}{D \log\left(\dfrac{D+2E}{D-2(l_a+e)}\right)}$$

$$C = \frac{\pi \beta B_e}{4 p B_{fer}} D, p = \frac{\pi D}{\Delta_p}, e_{min} - e \le 0, K_f - K_{f\max} \le 0$$

Example for the Dimensioning of an Electrical Motor

Electrical Slotless Rotating Machines with Permanent Magnet:



- IBBA standard (defined by Ratschek and Rokne 1988) $\longrightarrow$ 1h35,
- IBBA + propagation due to E. Hansen $\longrightarrow$ 41.5s,
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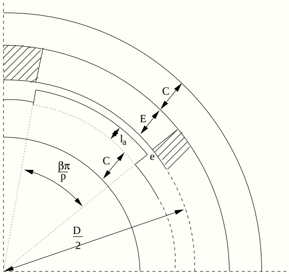

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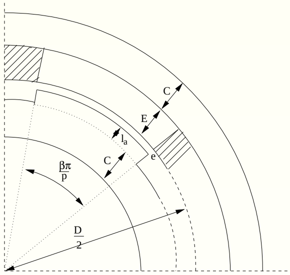

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Combination of Different Rotating Electrical Machines

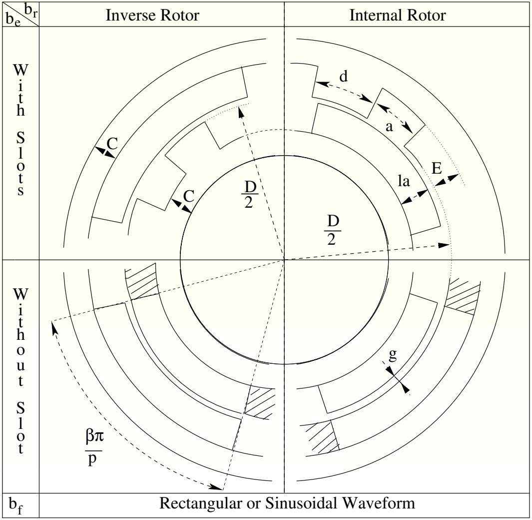


Figure: 4 structures possible machines $\times 2$ modes (rectangular or sinusoidal waveform).

Discrete Variables for Modeling Electrical Machines

1. $b_r = 1$ for machines with an internal rotoric configuration and $b_r = 0$ for an external one,
2. $b_e = 1$ for machines with slots or $b_e = 0$ slotless machines,
3. $b_f = 1$ represents rectangular waveform or $b_f = 0$ for a sinusoïdale one.

3 boolean variables to represent 8 possible structures + 2 categorical variables.

Combinatorial Models for Electrical Machines

$$
\begin{aligned}
\mathbf{\Gamma_{em}} &= \mathbf{k_\Gamma} D\left[D + (1-b_e)(2b_r-1)E\right] L\mathbf{B_e}\mathbf{K_S},\\
\mathbf{K_S} &= k_r Ej\left(b_e\frac{a}{a+d} + (1-b_e)\right),\\
\mathbf{k_\Gamma} &= \frac{\pi}{2}\left[b_f[1-\mathbf{K_f}]\sqrt{\beta} + (1-b_f)\frac{\sqrt{2}}{2}\sin(\beta\frac{\pi}{2})\right],\\
\mathbf{K_f} &= 1.5p\beta\left[\frac{E+g}{D}\right](1-b_e).b_f,\\
\mathbf{B_e} &= \frac{2\mathbf{J}(\sigma_\mathbf{m})l_a}{(2b_r-1)D\ln\left[\frac{D+2E(2b_r-1)(1-b_e)}{D-2(2b_r-1)[l_a+g]}\right]}\frac{1}{\mathbf{k_c}},\\
\mathbf{k_c} &= \frac{1}{1-b_e\left[\frac{N_e a^2}{5\pi D.g+\pi D.a}\right]},\\
&\dots
\end{aligned}
$$

Generally, the torque Γ_{em} is fixed $\Longrightarrow$ a strong equality

Examples of 4 optimal machines with magnetical effects

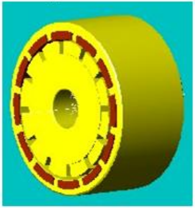
Min Volume of Magnets

- internal rotor
- rectangular wav.
- with slot
- mag. powders
- p = 2
- magnet NdFeB

Min Volume of active parts

- internal rotor
- sinusoidal wav.
- slotless
- sheet metal
- p = 6
- magnet NdFeB

Min Weight



- internal rotor
- sinusoidal wav.
- slotless
- mag. powders
- p = 6
- special-magnet

Min Global Volume

- internal rotor
- rectangular wav.
- with slot
- sheet metal
- p = 5
- magnet NdFeB

Numerical Validations

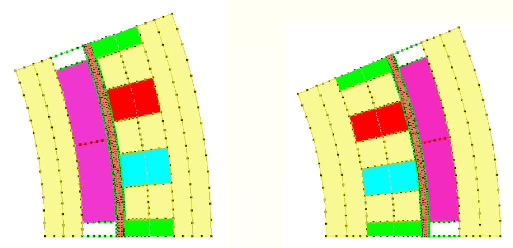
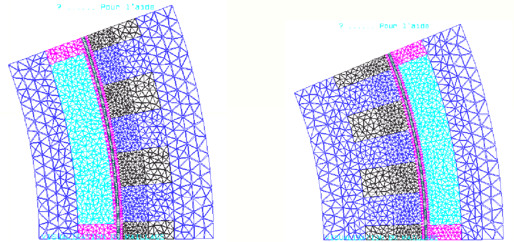
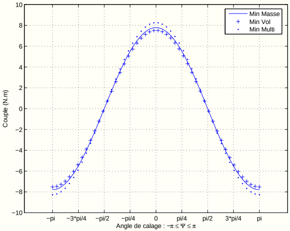


Figure: Draw 2 optimal solutions (min mass and min multicriteria).



Extension: Numerical Validations



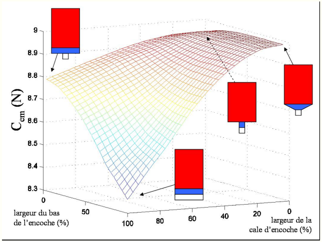


Figure: Torque of 3 solutions and design of teeth of the slot.

Using Triangle and EFCAD.
Name of the Software: NUMT.

Some Realizations

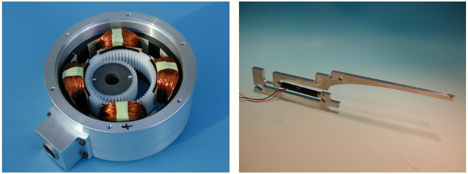


Figure: Motor with a strongest torque.

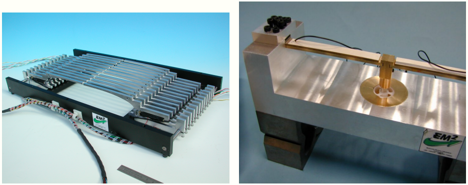


Figure: Design of piezoelectric bimorphs.

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